# Why expected loss is not the right measure of credit risk and how to do better*

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First draft: May 21, 2015
This draft: March 30, 2017

**Abstract**

The large bulk of risk assessment in the banking industry, including that for regulatory purposes, is performed using satellite models. These are founded on the conditional independence assumption, which focuses on the variability in balance sheet outcomes attributable to macro factors. The quantiles of a financial variable distribution are imputed indirectly from the quantiles of the factors. The volatility that is not explained by a Gaussian factor model is simply discarded via the *expected loss* formulation. I argue that the above results in a failure to model salient features of the joint distribution of the balance sheet outcomes and macroeconomic indicators, and results in misleading estimates of the quantiles of the financial loss distribution. I propose an alternative to satellite models, specifying the joint likelihood as a flexible construction of pairwise copulas. This allows direct inference on the true marginal (as opposed to the conditional) distribution of the financial variables of interest and produces more realistic estimates of its quantiles.

*Any views expressed are solely those of the author. This paper should therefore not be reported as representing the views of the Bank of England or members of the Monetary Policy Committee or Financial Policy Committee. I am grateful to Andrea Serafino for extensive discussions, and to Graham Mortimer for organising support with algorithm implementation. Preliminary and incomplete, please do not quote.

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# 1 Introduction

This paper is concerned with the use of the *expected loss* concept in applications such as credit risk management in the financial industry and the related regulatory stress tests. The idea behind both is that identifying hazardous financial contingencies ex ante should help devising precautionary measures and emergency planning. Methodologies used in credit risk assessment and stress testing have evolved over the years, converging on a set of common principles. These appear to have been driven by the data scarcity and a peculiar organisational division of labour between economists and risk specialists rather than the underlying statistical properties of the data.

As a result, instead of analyzing statistical properties of the balance sheet outcomes in an unconstrained specification, some really strong assumptions are made. One of the mainstays of the credit risk analysis is the ubiquitous decomposition of expected loss into PD, LGD, and EAD. A concise introduction into these concepts, informed primarily by the accounting (rather than the statistical) logic, is given in (Schuermann 2004). But is the theoretical construct of expected loss of relevance to risk management at all? Answering this question requires clarifying the ultimate objective of modelling credit losses.

In what follows, section 2 provides an overview of the standard modelling approach to credit risk, the implicit statistical assumptions, and their empirical adequacy. Section 3 reviews the main results of the statistical literature on the pairwise copula decomposition of multivariate density functions and proposes a parsimonious way to specify a joint distribution of financial and macroeconomic variables using *vines*, or equivalently, R-vine arrays. Section 4 provides empirical evidence, comparing the tail inferences in the scenario-based approach and the joint probabilistic approach. Section 5 concludes, suggesting that the expected loss approach is grossly inadequate for evaluating credit risk in realistic environments.

# 2 Expected loss as a portfolio risk measure

The purpose of the analytical risk assessment is to infer salient characteristics of the distributions of portfolio outcomes relevant for the decision-making of portfolio managers, other market participants, and regulators. More specifically, the relevant characteristics are usually those of the future distribution of portfolio profit and loss, which are transformations of a set of balance sheet variables. Due to the presumed statistical association between balance sheet outcomes, on the one hand, and various economic aggregates and market prices, on the other, the historical series of the latter variables may contain valuable statistical information about the former. Consequently, at the most general level, the appropriate object of the analysis is the joint distribution of all of the above.

In the currently prevalent analytical practice, however, credit risk assessment is instead split into two main stages. In the first stage a deterministic macroeconomic scenario is designed. This is a collection of paths of aggregate indicators produced by teams of economists without explicit consideration of the "risk factors" (such as write-off rates or default probabilities, or risky asset prices). In the second stage the impact of the scenario on portfolio outcomes is assessed via a suite of "satellite" risk models. These purport to capture the statistical association between the state of the macroeconomy at a point in time and the performance of financial portfolios (it is a subtle but important point that the association is specified as backward-looking, i.e. the leads of macroeconomic variables are assumed to be uninformative for the balance sheet outcomes). The output of these satellite models is predominantly stated in terms of the cumulative expected loss of a portfolio, conditional on the macro scenario.

I refer to this practice below as "standard approach". The unstated premise seems to be that the severity of the scenario translates into the severity of cumulative portfolio outcomes "approximately" one-for-one. But under what statistical assumptions would this equivalence hold?

## 2.1 The assumptions behind the standard approach

Denote the vector of observations of macroeconomic aggregates (also referred to as stress factors) by $S_t$. Denote the vector of the balance sheet outcomes by $Z_t$. These could in principle be observed loan performance measures, e.g. write-off rate, or unobservable variables such as a probability of default of a given loan. These are also referred to as risk factors and form inputs into an accounting model of profit and loss. First, note that teams of economists that design macroeconomic scenarios, do not have balance sheet realisations in their datasets. To assume that this does not lead to any loss of generality in scenario choice is equivalent to

**Assumption 1** *Macroeconomic aggregates are not affected by balance sheet outcomes.*

In the second stage the impact of the scenario on portfolio outcomes is assessed via a suite of "satellite" risk models. These postulate a relationship between the current macro environment and the mean of the loss distribution, and use it to compute expected loss conditional on any given scenario[1].

Whenever balance sheets behave differently than predicted by a satellite model within the sample, the discrepancies are treated as nuisance terms, akin to errors of measurement.

[1] Note that if the scenario is multi-period, then the expected per-period loss is computed conditional on the current element of the data filtration, so the sequence of the per period losses looks like

$$E\left( Z_t | \{S_\tau\}_{\tau \leq t} \right),$$

rather than conditional on the entire scenario path

$$E\left( Z_t | \{S_{T+k}\}_{k=1,..,H} \right).$$

Even though statistical properties of this "noise" could in principle be used for a more complete description of the potential financial outcomes, these are in most cases simply discarded, amounting to

**Assumption 2** *Balance sheet outcomes are driven solely by macroeconomic aggregates.*

There are occasional exceptions to the above extreme assumption, whereupon the satellite equations for multiple portfolios may be estimated as a system, and the variance-covariance matrix of residuals may be used for a slightly subtler analysis than the straightforward calculation of the aggregate expected loss would entail. In these instances the assumption of the Gaussian copula is ubiquitous.

**Assumption 3** *The joint distribution of macroeconomic variables and risk factors is Gaussian.*

## 2.2 The prima facie case for revisiting the expected loss paradigm

Since aggregate regressors in satellite models are assumed to be the only variables driving systematic variation in balance sheet outcomes, their selection is paramount for the accuracy of the estimation. But the statistical fit of most satellite models is quite poor. That is, the "noise" component is very large, many times the component explained by macro variables (Guerrieri and Welch 2012). This suggests that in empirical reality there may be important contributors to financial risk not captured in the macroeconomic indicators.

Secondly, the independence of macroeconomic shocks from risk factors is not supported by the historical evidence on financial crises (Borio, Drehmann and Tsatsaronis 2013). Thus the set of macroeconomic scenarios that can be considered does not allow for propagation and magnification of the shocks of financial origin, leading to further loss of generality.

Finally, multivariate normal distribution (and other distributions with non-Gaussian marginals based on Gauss copula) are in general easy to work with because most calculations can be performed in closed form. From the purely operational point of view this modelling choice thus has an important advantage of scalability, since the computational costs do not rise significantly even as as the dataset size gets substantial. Gauss copula, however, has special properties that make it overly restrictive when dealing with realistic datasets mixing macroeconomic, financial, and balance sheet data. One of its biggest issues is that it exhibits zero tail dependence. If we consider portfolio outcomes such as loss rate, default rate, non-performing loans, etc. this means that joint realisations of tail outcomes are extremely unlikely under the assumed data-generating process. In the context of credit risk, for example, this means that mass defaults would have an infinitesimally small probability of occurring even under extreme values of correlation parameters.

## 2.3 Analytical integrity concerns

If macroeconomic shocks don't fully explain financial crisis episodes (Borio et al. 2013), then crucial sources of variability in portfolio outcomes may get entirely overlooked in the standard industry and regulatory practice. In the expected loss calculation the impact of these sources is simply averaged, resulting in scenarios heavy on macroeconomic stress, but light on the financial stress. As a result, it is customary for the modellers to find that even severe macroeconomic scenarios produce variation in balance sheet outcomes far smaller than the actual movements during financial crisis episodes. Because of this, extreme scenario realisations are required to "elicit" even modest losses. On both sides of the fence, regulators' and regulated firms', "there is pressure down the line [. . . ]to generate acceptable risk numbers" (Dowd 2014). To conform to their management's expectation, macroeconomists specify implausible scenarios, while satellite model teams resort to heavy use of judgement. The informational content in the data is significantly diluted, while the very circumstances that may cause great financial distress go unaccounted for. It may be fair to speculate that risk managers and policymakers end up scrutinizing phantom threats and ignoring real ones.

## 2.4 Putting the standard approach to test

The distributional assumptions above constitute the weaknesses of the traditional approach primarily because their validity has never been explicitly tested. Indeed, they are impossible to test in the prevailing expected loss paradigm. However, all of them contribute to making the estimates of portfolio losses less conservative. It is therefore desirable to build a full-information probabilistic representation of the joint data-generating process for portfolio outcomes.

The advantage of the joint probabilistic approach to credit risk lies in explicitly acknowledging multiple sources of shocks to balance sheet variables that may not be limited to macroeconomic variables, and also in flexible estimation of the statistical properties of those shocks.

# 3 Overview of the vine-copula likelihood representation and parsimonious specification issues

The recent advances in statistics and uncertainty theory, namely, the literature on copulas and graphical models, such as vines, made it possible for the first time to specify the joint data-generating process in a very flexible way. The starting point is the concept of copula and the product decomposition of the multivariate likelihood function described by an R-vine.

## 3.1 Likelihood factorisation using bivariate copulas

The few known *multivariate* copula functions fall into two classes: the elliptical (including the Gaussian copula and the $t$-copula) and the Archimedean ones (e.g. multivariate Clayton and Gumbel). The Gaussian copula, as discussed in Section 2, is extremely restrictive because it does not allow any potential tail dependence present in the data to manifest itself. The $t$-copula, whilst permitting non-zero degree of tail dependence, is still overly restrictive. This is because the single parameter (degrees of freedom) governs the tail dependence across all of the variables[2], which limits the capability of the copula to express heterogeneous patters of interdependence across different classes of variables in a multivariate dataset (see, for instance, Luo and Shevchenko (2010)). The Archimedean multivariate copulas are also very restrictive because they are by construction typically parameterised by one, at most two parameters. It is therefore important on the prima facie grounds to allow for intertemporal as well as cross-sectional interdependence, which may manifest very differently.

The above discussion suggests that for credit risk applications a researcher should use a copula family that exhibits tail dependence at least for some of the parameter values, but flexibly across various constellations of variables. The issue of how much or how little tail dependence there is, is purely empirical – parameter estimates obtained from historical data should dictate that.

One is interested in the broad class of regular vines, or R-vines, which supply all possible bivariate copula decompositions of joint densities. Morales-Nápoles (2008) introduced the notion of an R-vine array, which without loss of generality encodes all regular vines, and hence allows one to bypass the details of the vine theory when considering all possible pairwise decompositions of the joint likelihood function.

| | | | | | | |
|---|---|---|---|---|---|---|
| $m_{1,1}$ | | | | | | |
| | $\ddots$ | | | | | |
| | | $m_{i,i}$ | | | | |
| | | | $m_{i+1,i+1}$ | | | |
| | | $m_{k,i}$ | | $\ddots$ | | |
| | | | | | $m_{N,2}$ | |
| $m_{N,1}$ | | $m_{N,i}$ | $m_{N,i+1}$ | | $m_{N,1}$ | $m_{N,1}$ |

Table 1: A generic R-vine array

Consider a lower-triangular matrix $(m_{i,j})_{\ i,j=1,..,N}$ with elements $m_{i,j} \in \{1,..,N\}$ Morales-Nápoles (2008) has shown that the matrix $(m_{i,j})_{\ i,j=1,..,N}$ corresponds to a regular

[2] In addition, the dependence is symmetric in the upper and the lower tails. If the underlying data generating process is not symmetric in this respect, then estimating parameters of the copula may confuse the inference from the two tails.

vine iff the so called proximity condition holds. Define

$$\begin{aligned} B_M(i) &= \{(m_{i,i}, D)|\, k = i+1,..,N; D = \{m_{k,i},..,m_{N,i}\}\}, \\ \tilde{B}_M(i) &= \{(m_{k,i}, D)|\, k = i+1,..,N; D = \{m_{i,i}\} \cup \{m_{k+1,i},..,m_{N,i}\}\} \end{aligned}$$

**Definition 1** *A lower triangular array $M$ is called an R-vine array if it satisfies the proximity condition, namely, for $i = 1, N-1$ and for all $k = i+1, ..., N-1$ there is $j$ in $\{i+1, ...N-1\}$ such that $(m_{k,i}, \{m_{k+1}, ..., m_{N,i}\}) \in B_M(j) \cup \tilde{B}_M(j)$.*

Intuitively the condition means that for any conditional distribution of the subset of variables there exists a single internally consistent representation in terms of lower order conditional distributions. Two corollaries of the proximity condition that may be useful for following the arguments of Section 3.3 follow.

**Corollary 1** $\{m_{i,i}, ..., m_{N,i}\} \subseteq \{m_{j,j}.., m_{N,j}\}$ *for* $1 \leq j < i \leq N$;

**Corollary 2** $m_{i,i} \notin \{m_{i+1,i+1}, ..., m_{N,i+1}\}$ *for* $1 \leq i \leq N-1$.

The information contained in the R-vine array is interpreted as follows: for $m_{k,i}$, $k > i$, $i = 1, ..N-1$, the product decomposition of the likelihood contains copula density $c_{m_{i,i}, m_{k,i}|m_{k+1,i},..,m_{N,i}}\left(F\left(x_{m_{i,i}}\middle| x_{m_{k+1,i}},..,x_{m_{N,i}}\right), F\left(x_{m_{k,i}}\middle| x_{m_{k+1,i}},..,x_{m_{N,i}}\right)\middle| x_{m_{k+1,i}},..,x_{m_{N,i}}\right)$. Notice that for $k < N$ the arguments of the bivariate copula are not the probability integral transforms $u_{m_{j,i}} = F\left(x_{m_{j,i}}\right)$ but conditionial marginals, with the conditioning set consisting of the elements of the dataset indexed by the elements of the R-vine array lying below $m_{k,i}$. For more information on the properties of R-vine arrays I refer the reader to Dissmann, Brechmann, Czado and Kurowicka (2012). Also note that in general the bivariate copulas entering the decomposition depend on the values of the conditioning variables $x_{m_{k+1,i}},..,x_{m_{N,i}}$ both directly and indirectly (through the conditional distribution functions serving as arguments). The current literature has not yet explored computationally tractable ways to implement the direct dependence, picking from the menu of off-the-shelf parametric families such as Frank, Gumbel, or Clayton. However, Hobæk Haff, Aas and Frigessi (2010) have recently assessed the empirical impact of this simplification and found that the simplified copula construction are rather good approximations to the exact constructions in most cases. Hence I keep with the current practice and omit the conditioning terms as direct arguments to the copula densities in the current implementation of the model.

In order to reconstruct the joint likelihood information using the vine (or the corresponding R-vine array) one needs to supply the bivariate copula densities $c_{m_{i,i}, m_{k,i}|m_{k+1,i},..,m_{N,i}}$ for all $m_{i,j}$, $1 \leq j < i < N$ and the set of continuous invertible marginal distribution functions $F_1(\cdot), ..., F_N(\cdot)$. Together, these form a *pairwise copula construction* $(M, \mathcal{C}, \mathbf{F})$.

**Theorem 1** *For a pairwise copula construction based on an R-vine array, the results of Bedford and Cooke (2002) and Morales-Nápoles (2008) jointly imply that that there is a* *unique joint distribution of* $x_1, ..., x_N$ *with the density*

$$f_{1,...,N} = \prod_{j=1}^{N} f_j \prod_{i=1}^{N-1} \prod_{k=i+1}^{N} c_{m_{i,i},m_{k,i}\left|m_{k+1,i},..,m_{N,i}\right.} \left(F_{m_{i,i}|m_{k+1,i},..,m_{N,i}}\left(x_{m_{i,i}}\middle| x_{m_{k+1,i}}, .., x_{m_{N,i}}\right),\right.$$
$$\left. F_{m_{k,i}\left|m_{k+1,i},..,m_{N,i}\right.}\left(x_{m_{k,i}}\middle| x_{m_{k+1,i}}, .., x_{m_{N,i}}\right)\right) \quad (1)$$

Furthermore, the arguments of the conditional copula functions can be computed using the following result by Joe (1996).

**Theorem 2** *For an arbitrary random set* $\{x_{n_0}, .., x_{n_m}\}$ *with* $m \geq 2$,

$$F_{n_0|n_1,..,n_m}\left(x_{n_0}\middle|\{x_{n_1}, .., x_{n_m}\}\right) = \left.\frac{\partial C_{n_0,n_1|n_2,...,n_m}(u_0,u_1)}{\partial u_1}\right|_{\substack{u_1 = F_{n_1|n_2,..,n_m}\left(x_{n_1}\middle|\{x_{n_2},..,x_{n_m}\}\right)\\ u_0 = F_{n_0|n_2,..,n_m}\left(x_{n_0}\middle|\{x_{n_2},..,x_{n_m}\}\right)}}. \quad (2)$$

It is customary and convenient to refer to construct in (2) as an $h$-function,

$$F_{n_0,n_1|n_2,...,n_m}\left(x_{n_0}\middle|\{x_{n_1}, .., x_{n_m}\}\right) = h_{n_0,n_1|n_2,...,n_m}\left(F\left(x_{n_0}\middle|\{x_{n_2}, .., x_{n_m}\}\right), F\left(x_{n_1}\middle|\{x_{n_2}, .., x_{n_m}\}\right)\right).$$

Note that since the independence copula is a product of its arguments $C^{IND}(u_0, u_1) = u_0u_1$, the corresponding $h$-function is

$$h^{IND}(u_0, u_1) = u_0$$

for any conditioning set and any arguments.

## 3.2 Applied model selection and parsimony

In applications, the selection of a model (pairwise copula construction) starts with the selection of the R-vine array. The literature on this selection is still in its infancy. Two categories of approaches have been proposed so far. However, both of them can be referred to as *dense* approaches to populating the copula array $\mathcal{C}$. Smith (2015) focuses specifically on modelling panel datasets. He advocates organising data in D-vines (with data grouped in blocks corresponding to the same period observations and organised chronologically). Dissmann et al. (2012), Czado (2010) and Kurowicka (2011) develop algorithms for selecting vine specifications by sequentially choosing the links to be parameterised based on their ranking by the sample Kendall's tau or the partial correlation coefficient. However, since for a given multivariate density $f_{1,...,N}$ any two R-vines that are part of pairwise copula constructions are equally valid, it is not yet clear what those heuristic criteria represent.

The similarity of the two classes of approaches lies in parameterising all bivariate copulas implied by the R-vine. An $n$-dimensional R-vine needs specifying $\frac{n(n-1)}{2}$ bivariate copula families and $n$ marginal distribution families to complete the pairwise copula construction. Both Smith (2015) and Dissmann et al. (2012) parameterise contiguous sub-blocks of R-vine by distinct and non-degenerate copulas (in other words, non-independence copulas). This creates a concern about the parsimony of the estimated model.

Clearly, positing an $n$-dimensional R-vine for an $n$-dimensional dataset is impractical. Smith (2015) deals with this by assuming that the dataset has Markov property of a finite order and estimating copulas in recurrent contiguous D-vine blocks. Dissmann et al. (2012) assume strong independence of multivariate realisations across time, in which case the size of the R-vine corresponds to the cross-section dimension. Nevertheless both approaches are agnostic and do not exploit any structural assumptions, e.g. on the symmetries across links among the variables. In this sense they are akin to estimating unrestricted reduced form vector autoregressive models on panel datasets. Ir order to deal with the curse of dimensionality, Smith (2015) employs a Bayesian stochastic model search technique based on hierarchical priors.

I propose a different approach, which posits that the copula specification array is sparse and may have several instances of the same copulas (which ones will depend on the dataset). I start by considering the implications of the traditionally made distributional assumptions in terms of the implied pairwise copula construction.

Consider a dataset traditionally used in applications of the Vasicek model, with a single stress factor $S_t$ creating intratemporal correlation across loss rates (also referred to as default intensities) of $N$ credit portfolios, indexed by $j$, $Z_{j,t}$. The matrix fragment in Table 2 specifies the R-vine for the dataset commencing at time 1 and only shows variables up to period 2 (the same patterns apply in all subsequent periods). The red rectangles denote the corresponding elements of the copula array that are posited as non-degenerate (non-independence). The rest of the interdependence links are indeed assumed to be missing. Furthermore, all the copulas linking $Z_{j,t}$ to $S_t$ are assumed to be instances of the same copula family (Gaussian with the parameter $\rho$), while copulas linking $S_t$ to $S_{t-1}$ ($S_1$ to unobserved $S_0$ constituting the special initialization instance) are all instances of the same Gaussian copula with the parameter $\alpha$ of the AR(1) process. Note that only two copula instances are postulated and estimated, whereupon Smith's (2015) approach would have allowed up to $\frac{3N^2+5N+1}{2}$ distinct copula instances for the Markov process of order 1.

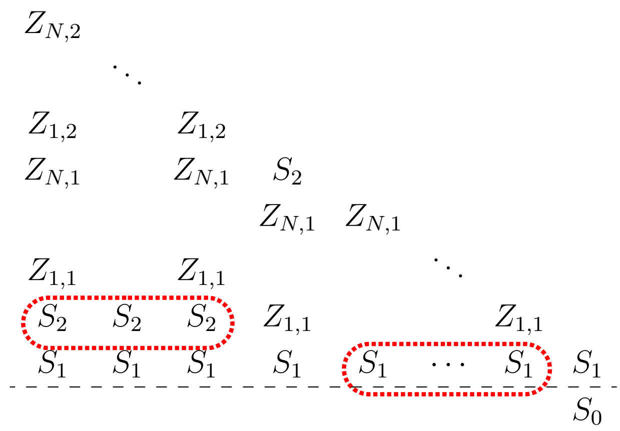


Table 2: Representation of a one factor i.i.d. Vasicek model by an R-vine array

The framework admits both specifications (Smith and Vasicek) as extreme special cases. It seems natural to explore the middle ground between the two approaches by relaxing conditional independence assumptions of the Vasicek model but keeping portfolio symmetries for the sake of parsimony. In particular, one would like to maintain the assumption that the contemporaneous links between the stress factor and the portfolio variables are symmetric (I will refer to them as satellite links in what follows, although there is absolutely no implication of causality running from stress factors to the credit outcomes). One might want to add further (symmetric) intratemporal links amongst the default intensities of different contracts, further (symmetric) intertempotal links across the same contract over time, and the feedback from default intensities to the stress factor next period (see Table 3), though I am not switching all of these links in the empirical example of Section 4.

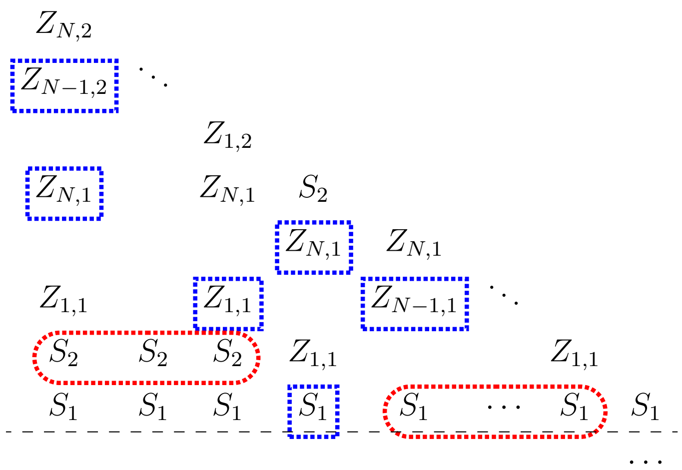


Table 3: One factor model augmented by macro driver persistence, intra- and intertemporal links and micro to macro feedback

It is clear from the above that pairwise copula constructions allow modelling structures of joint dependence that are both flexible and parsimonious. The specification in Table

3 can be extended to allow for more factors, higher Markov order across time, stronger conditional linkages both in the cross-section and in the time series dimension, as well as more complex feedbacks.

## 3.3 Statistical inference

### 3.3.1 Simulation of tail outcomes

The likelihood function allows an analyst to perform both frequentist (MLE) and Bayesian inference.

The frequentist approach, unlike the Bayesian, ignores the parameter uncertainty whenever it is present, artificially shrinking the support of the distribution of outcomes. By pretending that the model parameters, once estimated, are known with certainty, classical approach effectively distorts that distribution. The bigger the parameter uncertainty in the sample, the more severe is the distortion. If one is interested in conservative modelling of the extreme realisations as in stress scenario design one should be conscious not of the sampling uncertainty, but also the parameter uncertainty, which in finite sample may be substantial.

Thus in what follows we will be performing Bayesian integration over the posterior distribution of the parameters in order to construct a sample consistently representating the full joint distribution of the financial and macro variables.

### 3.3.2 Piece-wise vs. joint parameter inference

There are several ways of dealing with the estimation of the parameters of the marginal distributions and the copulas. A crude but popular option is to use the empirical marginal distributions (sample grades) and bypass the issue entirely. Another possibility is a two step procedure in which the parameters of the marginals are estimated first and the second step is performed conditionally on the first stage estimates. I opt for the joint Bayesian treatment of both sets of parameters because this is the only way to preserve the information contained in the sample. dos Santos Silva and Lopes (2008) provide the corroborating evidence that the model identification improves when the copula and marginal distribution parameters are jointly estimated.

### 3.3.3 Prior assumptions

I postulate the following diffuse priors for copula parameters: for every copula family their corresponding Kendall's $\tau$ is assumed to be distributed uniformly on $[\tau_{\min}, \tau_{\max}]$. The interval boundaries are the respectively the lowest and the highest possible values of $\tau$ for a given copula family. For Frank and Gauss copulas, for example, this range is $[-1, 1]$, while Gumbel and Clayton are limited to positive interdependence and the range is $[0, 1]$.The distribution of the copula parameter is implied by the bijection between the copula parameter and $\tau$. For marginal distributions I use flat priors over the support of the parameter, i.e. $(-\infty,+\infty)$ for the univariate normal mean and $[0,+\infty)$ for the variance.

### 3.3.4 Modified MCMC Kameleon algorithm

The main challenge when sampling from complex multivariate distributions (multimodal margins, disjoint support sets, etc.) using Metropolis-Hastings algorithm is computational efficiency, that is achieving convergence to ergodic chain without running excessively long simulations. Adaptive methods aim at increasing sampler's efficiency by utilising information in the history of the chain to alter the proposal distribution (Christophe and Thoms 2008). In practice, the more complex the distribution, the less effective simple adaptation strategies are. Sejdinovic, Strathmann, Garcia, Andrieu and Gretton (2014) have recently proposed an MCMC algorithm with superior mixing properties even for high dimensions

The proposal distribution $q(\cdot)$ is specific to the current chain state $y$. First a subsample $\mathbf{z}=\{z_j\}_{j=1}^{n}$ is randomly selected from the preceding chain history. Then for a given kernel $k(x,x'):\mathcal{X}\times\mathcal{X}\rightarrow\mathbb{R}$,the proposal is given by

$$
\begin{aligned}
q_{\mathbf{z}}(\cdot|\, y) &= N\left(y,\gamma^2 I+\nu^2 M_{\mathbf{z},y} H M'_{\mathbf{z},y}\right),\\
M_{\mathbf{z},y} &= 2\left[\nabla_x\, k(x,z_1)|_{x=y},...,\nabla_x\, k(x,z_n)|_{x=y}\right]
\end{aligned}
$$

Here $\gamma$ is an isotropic innovation scale, and $\nu$ is an adaptable scaling parameter for the reproduced kernel Hilbert space. Although a kernel can be any symmetric positive-definite function, the Gaussian kernel $k(x,x')=\exp\left(-\frac{\left\|x-x'\right\|_2^2}{2\sigma^2}\right)$ performs well in a wide range of situations.

This sampler works well when the parameters have no theoretical bounds. In the present case, whereupon a lot of parameters of copula density functions are restricted to an interval, and parameters of marginal distributions may be restricted to positive values, the implied proposal distribution parameters often lead to the simple inability to even propose a value that falls within the permitted range. A simple modification of the algorithm that restores its efficiency is transforming each coordinate domain $S_i$ by $g_i: S_i\rightarrow\mathbb{R}$. For example, logit function can be used to map variables from $[0,1]$ to the entire real line: the transformed state coordinate is computed as $\tilde{y}_i=-\ln\left(\frac{1}{y_i}-1\right)$, and if accepted, the proposed new coordinate is transformed back as $y_{i+1}=\left(1+e^{-\tilde{y}_{i+1}}\right)^{-1}$. The overall transformation is a Cartesian product: $g(y)=\times_i g_i(y_i)$. The proposal acceptance probability is modified to account for the Jacobian of the transformation:

$$
A(y,y^*)=\min\left\{1,\frac{\pi(y^*)\,q_{\mathbf{z}}(g(y)|\,g(y^*))\,\Pi_i\,\frac{dg_i}{dx_i}(x_i)\Big|_{x_i=y_i^*}}{\pi(y)\,q_{\mathbf{z}}(g(y^*)|\,g(y))\,\Pi_i\,\frac{dg_i}{dx_i}(x_i)\Big|_{x_i=y_i}}\right\}
$$

The adaptation consists of modifying parameter $\nu$ to coerce the acceptance probability a multivariate innovation to 0.234. This corresponds to the global adaptive approach in the vanishing adaptation framework of Christophe and Thoms (2008), and ensures the ergodicity of the produced chain. Various sampler diagnostics show that the modified kernel-adaptive Metropolis-Hastings algorithm results in very good effective sample sizes and low serial correlation, compared to alternative non-adaptive, and component-wise adaptive samplers.

For each parameter sample from the joint posterior I simulate forward projections of all the variables of interest. This allows me integrate numerically over the space of unknown model parameters, portfolio variables, and macroeconomic variables.

# 4 Empirical results

For the empirical illustration of the difference the joint probabilistic approach makes I use Greece dataset, consisting of various macro and market series, as well as annual provision rates (as proxies for loss rates) of four major Greek banks, obtained from S&P Capital IQ. The data frequency is annual, and the provision rate data exist up to 1990. I use the sample data up to 2014 to jointly project macroeconomic indicators and loss rates for three years ahead (2015-2017). The dimensionality of the state space is estimated to be 6 by the criterion of Bai and Ng. The latent macro factor series are recovered using the GRASTA algoritm, optimizing with respect to the linear subspace by minimising the robust $l_1$ norm on the Grassmannian (He, Zhang, Balzano and Tao 2014).

## 4.1 Expected loss

To emulate the output of the standard Vasicek model I posit a Gaussian AR(1) process for the single factor variable that explains the highest proportion of the variance in macro indicators. The factor variable is linked to the loss rates by a Gauss copula as in Table 2. I select three macroeconomic scenarios corresponding to the extreme percentiles of the cumulative GDP fall over the 3 years. For each of the three scenarios aggregate credit outcomes are calculated as conditional expected losses. The realisations of the GDP drop and aggregate loss rate are presented in Table 4. The impact of even the scenarios that feature extremely severe GDP drop appears to be minuscule.

| | Percentile | | |
|---|---|---|---|
| | 90% | 95% | 99% |
| GDP cumulative drop | -38.7% | -47.5% | -64.7% |
| Aggr. conditional cumulative loss rate | 1.877% | 1.878% | 1.880% |

Table 4: Conditional expected loss rate in one factor model

## 4.2 Unconditional loss distribution

As the alternative to the expected loss, I recover percentiles of the unconditional cumulative loss distribution under several different specifications of the vine-copula model. I experiment with the number of factors, the assumption on whether the vector of factors follows an autoregressive process or is i.i.d., and with the copula family for the satellite links. Table 5 reports the three upper tail percentiles of this distribution.

| Specification | 90% | 95% | 99% |
|---|---|---|---|
| 1 factor, AR(0), Gauss copula | 3.1% | 3.4% | 4.0% |
| 1 factor, AR(0), Clayton copula | 3.0% | 3.3% | 3.9% |
| 1 factor, AR(1), Gauss copula | 3.1% | 3.4% | 4.0% |
| 1 factor, AR(1), Clayton copula | 3.0% | 3.3% | 3.9% |
| 2 factors, VAR(1), Gauss copula | 3.1% | 3.5% | 4.1% |
| 2 factors, VAR(1), Clayton copula | 3.0% | 3.3% | 3.9% |
| 6 factors, VAR(0), Clayton copula | 3.6% | 4.2% | 5.4% |
| 6 factors, VAR(1), Gauss copula | 9.7% | 10.6% | 12.6% |
| 6 factors, VAR(1), Clayton copula | 10.3% | 11.7% | 14.7% |

Table 5: Percentiles of the unconditional loss rates

It appears that the 1st two factors are largely independent of the credit losses, at least when only the direct contemporaneous association is allowed by the specification. This is true regardless of the autoregressive specification of the factor process and of the copula family assumed to govern the satellite links. However, if one uses all 6 factors, the results are different. The most conservative estimates of the loss percentiles are for 6 autocorrelated factor model (the last two rows). The fact that between these two, assumption of Clayton satellite links results in more severe outcomes seems consistent with the discussion of the tail dependence properties.

Figure 1 plots densities of the cumulative distribution of aggregate loss rates of the Greek banking system over 3 years. The "Vasicek density" in the chart is in fact four separate densities discussed in Section 4.1, which are visually indistinguishable. It can be seen that the conditional means of the loss rate are highly insensitive to the scenario and close to the mean of the unconditional distribution.This should not be surprising, because the stress factor was chosen with no regard for the balance sheet variables solely on the basis of a state space model for the macroeconomic indicators. This reflects the typical practice in the industry.

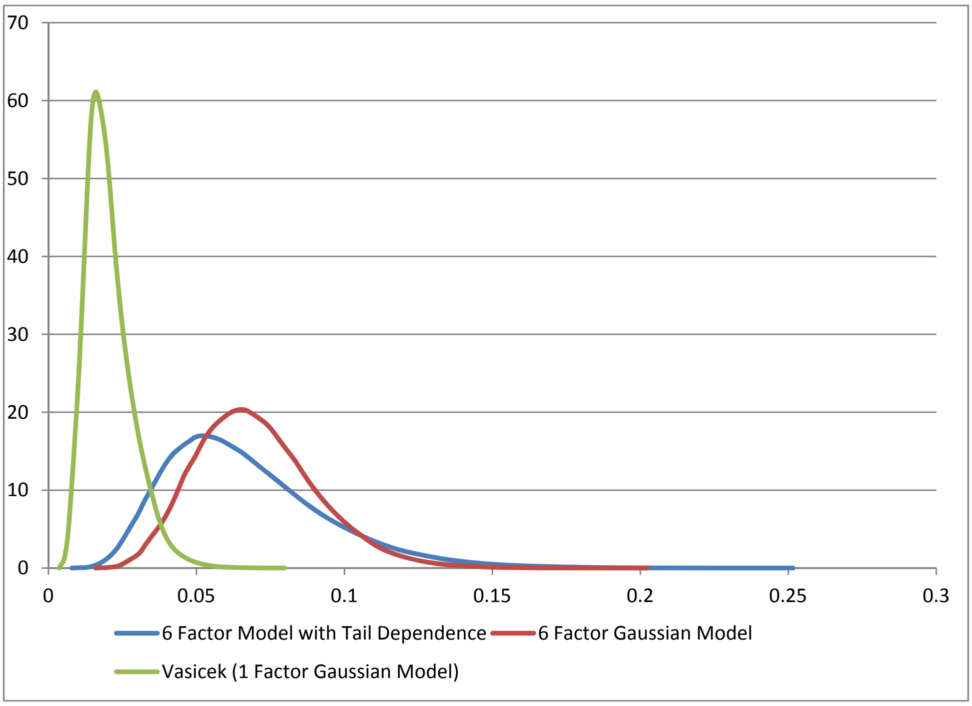


Figure 1: Conditional and unconditional densities of the cumulative aggregate loss rate

## 4.3 Loss-driven scenarios

If in the macroeconomic scenario paradigm one computes losses conditional on macro outcomes, it is possible to reverse this and consider macro outcomes conditional on a scenario defined in terms of loss distribution quantiles. Table 6 presents results from two model specifications, both with 6 factors following VAR(1), the Gaussian Markov process of order 1. What is different, is the copula families for the satellite links. The top left quarter shows loss rates corresponding to the *percentiles of the marginal distribution of losses* (top line) and *the marginal distribution of cumulative GDP drop* (bottom line). Note the marked difference, especially at the extreme tail values. The top right quarter shows corresponding statistic of GDP. One can see that a much more severe scenario in terms of GDP fails to “drive” comparable losses. This outcome is of course trivial and model-free, as long as losses and GDP are not deterministic function of one another, but the model captures the strength of the statistical association between them. Percentiles of one margin of a multivariate distribution are by no means equivalent to the percentiles of another margin.

The bottom quarters show the respective results for the model specification with Clayton satellite links. The difference between the two specification in the extent of conservatism was already discussed. One should not fail to note, that the copula specification allowing for lower tail dependence (Clayton), does not require as extreme (or extremely implausible) realisations of GDP to “drive” substantial credit losses.

| | Aggr. cum. loss rate | | | GDP cumulative drop | | |
|---|---|---|---|---|---|---|
| Model, principal margin | 90% | 95% | 99% | 90% | 95% | 99% |
| Gauss, loss rate | 9.7% | 10.6% | 12.6% | -14.9% | -15.5% | -16.8% |
| Gauss, GDP drop | 7.9% | 7.9% | 8.3% | -19.0% | -20.8% | -24.3% |
| Clayton, loss rate | 10.3% | 11.7% | 14.7% | -3.8% | -4.3% | -6.2% |
| Clayton, GDP drop | 8.3% | 8.6% | 9.4% | -7.2% | -8.9% | -12.3 |

Table 6: Choice of the principal variable to sort scenarios on and the associated scenario outcomes

The specifications used do not encompass all the possible inter-dependence links, however, they help demonstrating how switching back on various pairwise links improves the ability of the model to generate "realistic" tail outcomes. Future implementation will concentrate on adding intertemporal and intratemporal links between portfolio variables, and intertemporal feedbacks from the portfolio variables to the macro variables.

# 5 Conclusion

This paper presented an alternative framework for specifying joint data-generating process for balance sheet and macroeconomic outcomes, based on vine-copula likelihood decompositions. It allows studying joint distributions of micro (balance sheet) and macro data without relying on extreme conditional independence assumptions, and unencumbered by the straightjacket of the multivariate Gaussian copula. The conditional expected loss inference ("the standard approach") is replicated within the framework as special case, allowing to explicitly query the validity of the strong assumptions behind it. The empirical example suggests that severe macroeconomic scenarios may not be associated with tail realizations in credit portfolios' loss rates.

The standard approach to credit risk assessment and stress testing thus appears to be based on unrealistic assumptions about the joint distribution of balance sheet outcomes and macroeconomic aggregates, which endanger its ex-ante objective of identifying financial fragilities. Without the ability to adequately capture the stochastic properties of multivariate distributions (with special emphasis on the tails), the severe but plausible macroeconomic scenarios are rarely associated with tail realizations in banks' loss rates. As a consequence, it is nearly impossible to detect any real fragility in portfolios using the standard approach.

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